Volumes of Arithmetic Okounkov Bodies

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Contents

1   Introduction

This paper is an improvement and simplification of Yuan [Yu]. In particular, by a simpler method, it proves the identity in [Yu, Theorem A] without taking the limit $p \to \infty$. The method also simplifies the proofs of Moriwaki [Mo], where more general arithmetic linear series were treated.

Recall that [Yu] explored a way to construct arithmetic Okounkov bodies from an arithmetic line bundle, inspired by the idea of Okounkov [Ok1, Ok2] and Lazarsfeld–Mustaţă [LM] in the geometric case. Theorem A of [Yu] asserts that the volumes of the Okounkov bodies approximate the volume of the arithmetic line bundle. The main result of this paper asserts an exact identity before taking the limit.

On the other hand, Boucksom–Chen [BC] initiated a different way to construct Okounkov bodies in the arithmetic setting. From the proof of the main result in this paper, it is easy to recognize the similarity of the constructions in [BC] and in [Yu]. We will make a comparison at the end of this paper.

In the following, we recall the construction of [Yu] and state our main result. We use exactly the same notations as in [Yu] throughout this paper.

Volume of an arithmetic line bundle

Let X be an arithmetic variety of dimension d, i.e., a d-dimensional integral scheme, projective and flat over $\mathrm{Spec}(\mathbb{Z})$. For any hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \| \cdot \|)$ over X, denote

$$\widehat{H}^0(X, \overline{\mathcal{L}}) = \{s \in H^0(X, \mathcal{L}) : \|s\|_{\sup} \leq 1\}$$

and

$$\hat{h}^0(X, \overline{\mathcal{L}}) = \log \#\widehat{H}^0(X, \overline{\mathcal{L}}).$$

Define the volume to be

$$\mathrm{vol}(\overline{\mathcal{L}}) = \limsup_{m \to \infty} \frac{\hat{h}^0(X, m\overline{\mathcal{L}})}{m^d/d!}.$$

Recall that a line bundle $\overline{\mathcal{L}}$ is said to be *big* if $\mathrm{vol}(\overline{\mathcal{L}}) > 0$.

Note that Chen [Ch] proved that "limsup=lim" in the definition of $\mathrm{vol}(\overline{\mathcal{L}})$. The result was also obtained in [Yu] by the construction of the Okounkov body.

Arithmetic Okounkov Body

Assume that X is normal with smooth generic fiber $X_{\mathbb{Q}}$. Denote by $X \to \mathrm{Spec}(O_K)$ the Stein factorization of $X \to \mathrm{Spec}(\mathbb{Z})$. Then K is the algebraic closure of $\mathbb{Q}$ in the function field of X. For any prime ideal $\wp$ of O_K, denote by $\mathbb{F}_{\wp} = O_K/\wp$ the residue field, and by $N_{\wp}$ the cardinality of $\mathbb{F}_{\wp}$.

Let

$$X \supset Y_1 \supset \cdots \supset Y_d$$

be a flag on X, where each Y_i is a regular irreducible closed subscheme of codimension i in X. Assume that Y_1 is the fiber $X_{\mathbb{F}_{\wp}}$ of X above some prime ideal $\wp$ of O_K, and $Y_d \in Y_1(\mathbb{F}_{\wp})$ is a rational point.

Define a valuation map

$$\nu_{Y.} = (\nu_1, \cdots, \nu_d) : H^0(X, \mathcal{L}) - \{0\} \to \mathbb{Z}^d$$

with respect to the flag $Y.$ as follows. For any nonzero $s \in H^0(X, \mathcal{L})$, we first set $\nu_1(s) = \mathrm{ord}_{Y_1}(s)$. Let s_{Y_1} be a section of the line bundle $\mathcal{O}(Y_1)$ with zero locus Y_1. Then $s_{Y_1}^{\otimes(-\nu_1(s))} s$ is nonzero on Y_1, and let $s_1 = \left(s_{Y_1}^{\otimes(-\nu_1(s))} s \right)\Big|_{Y_1}$ be

the restriction. Set $\nu_2(s) = \text{ord}_{Y_2}(s_1)$. Continue this process on the section s_1 on Y_2, we can define $\nu_3(s)$ and thus $\nu_4(s), \cdots, \nu_d(s)$.

For any hermitian line bundle $\overline{\mathcal{L}}$ on X, denote

$$v_{Y.}(\overline{\mathcal{L}}) = \nu_{Y.}(\widehat{H}^0(X, \overline{\mathcal{L}}) - \{0\})$$

to be the image in $\mathbb{Z}^d$. The *arithmetic Okounkov body* $\Delta_{Y.}(\overline{\mathcal{L}})$ of $\overline{\mathcal{L}}$ is defined to be the closure of $\Lambda_{Y.}(\overline{\mathcal{L}}) = \bigcup_{m \geq 1} \frac{1}{m^d} v_{Y.}(m\overline{\mathcal{L}})$ in $\mathbb{R}^d$. It is a bounded convex subset of $\mathbb{R}^d$ if non-empty. The following is the main result of this paper.

Theorem 1.1. *If $\overline{\mathcal{L}}$ is big, then*

$$\text{vol}(\Delta_{Y.}(\overline{\mathcal{L}})) \log N_\wp = \frac{1}{d!} \text{vol}(\overline{\mathcal{L}}).$$

In [Yu], the author restricted to the case $\mathbb{F}_\wp = \mathbb{F}_p$, and could only prove that the left-hand side converges to the right-hand side as $p \to \infty$. The problem was caused by a large accumulated error term in the estimation. The solution of this problem is found by the author in the preparation of [YZ], a recent joint work of the author with Tong Zhang. The key is Theorem 2.1, which puts the filtration together and makes a more accurate control of the error term.

2 Lattice points in a filtration

In this section, we state and prove the result on lattice points in Theorem 2.1. It is the key for Theorem 1.1.

Fix a number field K. By a *normed O_K-module*, we mean a pair $(M, \{\|\cdot\|_\sigma\}_\sigma)$ consisting of a locally free O_K-module M of finite rank, and a collection $\{\|\cdot\|_\sigma\}_\sigma$ of $\mathbb{C}$-norms $\|\cdot\|_\sigma$ on $M \otimes_\sigma \mathbb{C}$, indexed by $\sigma : K \hookrightarrow \mathbb{C}$ and invariant under the complex conjugation.

Let $\overline{M} = (M, \{\|\cdot\|_\sigma\}_\sigma)$ be normed O_K-module. Define

$$\widehat{H}^0(\overline{M}) = \{m \in M : \|m\|_\sigma \leq 1, \quad \forall \sigma\},$$

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and
$$\hat{h}^0(\overline{M}) = \log \#\widehat{H}^0(\overline{M}).$$

Denote by
$$O_K\langle \widehat{H}^0(\overline{M})\rangle, \quad \mathbb{Z}\langle \widehat{H}^0(\overline{M})\rangle$$

respectively the O_K-submodule and the $\mathbb{Z}$-submodule of M generated by $\widehat{H}^0(\overline{M})$. For any $\alpha \in \mathbb{R}$, denote
$$\overline{M}(\alpha) = (M, \{e^{-\alpha}\|\cdot\|_\sigma\}_\sigma).$$

For a normed O_K-module $\overline{L}$ of rank one, define
$$\widehat{\deg}(\overline{L}) = \log\#(L/sO_K) - \sum_\sigma \log\|s\|_\sigma.$$

Here $s \in L$ is any non-zero element, and the definition is independent of the choice of s. It is just the usual arithmetic degree of a hermitian line bundle over $\mathrm{Spec}(O_K)$.

It is clear that the dual $\overline{L}^\vee$ gives a natural normed O_K-module of rank one, and that the tensor product $\overline{M} \otimes \overline{L}$ is a natural normed O_K-module. The identity of the tensor is given by the trivial normed O_K-module $\overline{O}_K = (O_K, \{|\cdot|_\sigma\}_\sigma)$, where $|\cdot|_\sigma$ is just the usual absolute value.

In some literatures, a normed O_K-module is also called a metrized vector bundle over $\mathrm{Spec}(O_K)$. A normed O_K-module of rank one is also called a metrized line bundle over $\mathrm{Spec}(O_K)$, or just a hermitian line bundle over $\mathrm{Spec}(O_K)$.

The innovation of this paper is the application of the following result.

Theorem 2.1. *Let K be a number field. Let $\overline{M}$ be a normed O_K-module, and $\overline{L}_0, \overline{L}_1, \cdots, \overline{L}_n$ be a sequence of normed O_K-modules of rank one. Here $\overline{L}_0$ is the trivial normed O_K-module of rank one. Denote*
$$\alpha_i = \widehat{\deg}(\overline{L}_i), \quad r_i = \mathrm{rank}_{O_K} O_K\langle \widehat{H}^0(\overline{M} \otimes \overline{L}_i^\vee)\rangle, \quad i = 0, \cdots, n.$$

Assume that
$$0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n.$$

Then there is a constant $C > 0$ depending only on K such that
$$\widehat{h}^0(\overline{M}) \geq \sum_{i=1}^n r_i(\alpha_i - \alpha_{i-1}) - C(r_0 \log r_0 + r_0),$$
$$\widehat{h}^0(\overline{M}) \leq \widehat{h}^0(\overline{M} \otimes \overline{L}_n^\vee) + \sum_{i=1}^n r_{i-1}(\alpha_i - \alpha_{i-1}) + C(r_0 \log r_0 + r_0).$$

The case $K = \mathbb{Q}$ is equivalent to [YZ, Proposition 2.3], which follows from the successive minima of Gillet–Soulé [GS]. The extension to the general K is provided by Huayi Chen in a private communication to the author. We start with a simple lemma in the following.

Lemma 2.2. *Denote $\kappa = [K : \mathbb{Q}]$ and denote by D_K the discriminant of K.*

(1) For any hermitian normed O_K-module $\overline{L}$ of rank one, we have

$$\hat{h}^0(\overline{L}) > \widehat{\deg}(\overline{L}) - \kappa \log 2 - \frac{1}{2} \log |D_K|.$$

(2) There is a constant $\delta \geq 0$ depend only on K such that

$$\mathrm{rank}_{O_K} O_K \langle \widehat{H}^0(\overline{M}(-\delta)) \rangle \leq \frac{1}{\kappa} \mathrm{rank}_{\mathbb{Z}} \mathbb{Z} \langle \widehat{H}^0(\overline{M}) \rangle \leq \mathrm{rank}_{O_K} O_K \langle \widehat{H}^0(\overline{M}) \rangle$$

for any normed O_K-module $\overline{M}$.

Proof. The result in (1) follows from Minkowski's theorem. The second inequality of (2) follows from the inclusion

$$\mathbb{Z} \langle \widehat{H}^0(\overline{M}) \rangle \subset O_K \langle \widehat{H}^0(\overline{M}) \rangle.$$

For the first inequality of (2), fix a basis $(a_1, \cdots, a_\kappa)$ of O_K over $\mathbb{Z}$. Set

$$\delta = \max\{\log |a_i|_\sigma : \ i = 1, \cdots, \kappa, \ \sigma : K \hookrightarrow \mathbb{C}\}.$$

Then

$$
\begin{aligned}
&O_K \langle \widehat{H}^0(\overline{M}(-\delta)) \rangle \\
\subset\ & \mathbb{Z} \langle a_1 \widehat{H}^0(\overline{M}(-\delta)), \cdots, a_\kappa \widehat{H}^0(\overline{M}(-c)) \rangle \\
\subset\ & \mathbb{Z} \langle \widehat{H}^0(\overline{M}) \rangle.
\end{aligned}
$$

The inequality also follows.

□

Proof of Theorem 2.1. If $K = \mathbb{Q}$, then $\overline{L}_i$ is isometric to $\overline{\mathbb{Z}}(\alpha_i)$, and thus

$$\overline{M} \otimes \overline{L}_i^\vee = \overline{M}(-\alpha_i).$$

The theorem is equivalent to the inequalities for $\hat{h}^0$ in [YZ, Proposition 2.3]. Next we consider the general case.

Denote $\kappa = [K : \mathbb{Q}]$ and $c = \log 2 + \dfrac{1}{2\kappa} \log |D_K|$. Denote $\beta_i = \alpha_i/\kappa$ for convenience. View $\overline{M}$ as a normed $\mathbb{Z}$-module, and apply [YZ, Proposition 2.3] to the sequence

$$0 = \beta_0 \leq \beta_1 + c \leq \cdots \leq \beta_n + c.$$

We obtain

$$\widehat{h}^0(\overline{M}) \ \leq \ \widehat{h}^0(\overline{M}(-\beta_n - c)) + \sum_{i=1}^{n} r'_{i-1}(\beta_i - \beta_{i-1}) + O(r_0 \log r_0).$$

Here

$$r'_i = \mathrm{rank}_{\mathbb{Z}} \mathbb{Z} \langle \widehat{H}^0(\overline{M}(-\beta_i - c)) \rangle, \quad i = 0, \cdots, n-1.$$

We claim that $r'_i \leq \kappa r_i$ by our special choice of c. It simply implies the second equality we need to prove. By Lemma 2.2 (1), $\widehat{H}^0(\overline{L}_i^{\vee}(\beta_i + c)) \neq 0$. Since

$$\overline{M} \otimes \overline{L}_i^{\vee} = \overline{M}(-\beta_i - c) \otimes \overline{L}_i^{\vee}(\beta_i + c),$$

we have

$$r'_i = \mathrm{rank}_{\mathbb{Z}} \mathbb{Z} \langle \widehat{H}^0(\overline{M}(-\beta_i - c)) \rangle \leq \mathrm{rank}_{\mathbb{Z}} \mathbb{Z} \langle \widehat{H}^0(\overline{M} \otimes \overline{L}_i^{\vee}) \rangle \leq \kappa r_i.$$

The last inequality follows from Lemma 2.2 (2). It proves the claim.

As for the first inequality, apply [YZ, Proposition 2.3] to $\overline{M}(\delta + c)$ viewed as a normed $\mathbb{Z}$-module, and the sequence

$$0 = \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n.$$

Here δ is as in Lemma 2.2 (2). We obtain

$$\widehat{h}^0(\overline{M}(\delta + c)) \geq \sum_{i=1}^{n} r''_i(\beta_i - \beta_{i-1}) + O(r_0 \log r_0 + r_0).$$

Here

$$r''_i = \mathrm{rank}_{\mathbb{Z}} \mathbb{Z} \langle \widehat{H}^0(\overline{M}(-\beta_i + \delta + c)) \rangle.$$

Lemma 2.2 (1) gives $\widehat{H}^0(\overline{L}_i(-\beta_i + c)) \neq 0$, and thus

$$r''_i \geq \mathrm{rank}_{\mathbb{Z}} \mathbb{Z} \langle \widehat{H}^0(\overline{M} \otimes \overline{L}_i^{\vee}(\delta)) \rangle \geq \kappa r_i.$$

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Here the last inequality follows from Lemma 2.2 (2). The proof is finished by the basic inequality

$$\widehat{h}^0(\overline{M}) \geq \widehat{h}^0(\overline{M}(\delta + c)) - \kappa r_0(\delta + c + \log 3).$$

See [YZ, Proposition 2.1] or the original form [GS, Proposition 4]. $\qquad\square$

3 The arithmetic Okounkov body

In this section, we prove Theorem 1.1, and compare our construction with that of [BC].

Proof of the main theorem

Recall that X is a normal arithmetic variety with smooth generic fiber, geometrically irreducible over O_K. Recall that we have the flag

$$X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$$

of regular vertical subvarieties. We have assume that $Y_1 = X_{\mathbb{F}_\wp}$ for some prime ideal $\wp$ of O_K, and $Y_d \in Y_1(\mathbb{F}_\wp)$ is a rational point.

By [Yu, Proposition 2.5], if $\overline{\mathcal{L}}$ is a big hermitian line bundle on X,

$$\lim_{m \to \infty} \frac{\# v_{Y.}(m\overline{\mathcal{L}})}{m^d} = \mathrm{vol}(\Delta_{Y.}).$$

Hence, Theorem 1.1 is reduced to the following result, which strengthens [Yu, Theorem 2.6].

Theorem 3.1. *For any* $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(X)$,

$$\lim_{m \to \infty} \left| \frac{\# v_{Y.}(m\overline{\mathcal{L}})}{m^d} \log N_\wp - \frac{\widehat{h}^0(X, m\overline{\mathcal{L}})}{m^d} \right| = 0.$$

Now we prove the theorem. Recall that

$$v(\overline{\mathcal{L}}) = v_{Y.}(\overline{\mathcal{L}}) = \nu_{Y.}(\widehat{H}^0(X, \overline{\mathcal{L}}))$$

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is the image in $\mathbb{Z}^d$ of the valuation map

$$\nu = \nu_{Y.} = (\nu_1, \cdots, \nu_d) : H^0(X, \mathcal{L}) - \{0\} \to \mathbb{Z}^d$$

defined by the flag

$$X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d.$$

The flag

$$Y_1 \supset Y_2 \supset \cdots \supset Y_d$$

on the ambient variety Y_1 induces a valuation map

$$\nu^\circ = (\nu_2, \cdots, \nu_d) : H^0(X, \mathcal{L}_{\mathbb{F}_\wp}) - \{0\} \to \mathbb{Z}^{d-1}$$

of dimension $d - 1$ in the geometric case. Similarly, we have a valuation map ν° on the line bundle $(\mathcal{L} \otimes \wp^i)|_{Y_1}$ for any $i \in \mathbb{Z}$. There are natural isomorphisms $(\mathcal{L} \otimes \wp^i)|_{Y_1} \cong \mathcal{L}_{\mathbb{F}_\wp}$, which are compatible with the valuations.

The valuations ν and ν° are compatible in the sense that

$$\nu(s) = \left(\nu_1(s), \nu^\circ((s_{Y_1}^{\otimes(-\nu_1(s))}s)|_{Y_1})\right), \quad s \in \widehat{H}^0(X, \overline{\mathcal{L}}) - \{0\}.$$

Here s_{Y_1} is the section of $\mathcal{O}(Y_1)$ defining Y_1.

Decompose $v(\overline{\mathcal{L}}) = \nu_{Y.}(\widehat{H}^0(\overline{\mathcal{L}}))$ according to the first component in $\mathbb{Z}^d$. We have

$$v(\overline{\mathcal{L}}) = \coprod_{i \geq 0} \left(i, \ \nu^\circ(\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)|_{Y_1})\right).$$

Here $\wp$ is naturally a hermitian line bundle on $\mathrm{Spec}(O_K)$, endowed with the metric induced from the trivial metric of O_K via the inclusion $\wp \subset O_K$. It is also viewed as a hermitian line bundle on X by pull-back.

By the geometric case in [LM],

$$\#\nu^\circ(\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)|_{Y_1}) = \dim_{\mathbb{F}_\wp} \mathbb{F}_\wp\langle\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)|_{Y_1}\rangle.$$

Here $\mathbb{F}_\wp\langle\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)|_{Y_1}\rangle$ denotes the $\mathbb{F}_\wp$-vector subspace of $H^0(\mathcal{L} \otimes \wp^i|_{Y_1})$ generated by $\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)|_{Y_1}$. The reduction map gives an injection

$$H^0(\mathcal{L} \otimes \wp^i)/\wp H^0(\mathcal{L} \otimes \wp^i) \longrightarrow H^0(\mathcal{L} \otimes \wp^i|_{Y_1}).$$

It follows that

$$\dim_{\mathbb{F}_\wp} \mathbb{F}_\wp\langle\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)|_{Y_1}\rangle = \mathrm{rank}_{O_K} O_K\langle\widehat{H}^0(\overline{\mathcal{L}} \otimes \wp^i)\rangle.$$

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Here $O_K\langle\widehat{H}^0(\overline{\mathcal{L}}\otimes\wp^i)\rangle$ denotes the O_K-submodule of $H^0(\mathcal{L}\otimes\wp^i)$ generated by $\widehat{H}^0(\overline{\mathcal{L}}\otimes\wp^i)$. Therefore,

$$\#v(\overline{\mathcal{L}})=\sum_{i\geq0}\mathrm{rank}_{O_K}O_K\langle\widehat{H}^0(\overline{\mathcal{L}}\otimes\wp^i)\rangle. \tag{1}$$

Note that it is essentially a finite sum.

For any $i\geq0$, denote $\overline{L}_i=\wp^{-i}$. Then

$$\widehat{\deg}(\overline{L}_i)=i\log N_\wp.$$

Apply Theorem 2.1 to the normed O_K-module

$$\overline{M}=(H^0(\mathcal{L}),\{\|\cdot\|_{\sigma,\sup}\}_\sigma)$$

and the sequence $\{\overline{L}_i\}_{i\geq0}$. It is easy to obtain

$$\widehat{h}^0(\overline{\mathcal{L}})=\sum_{i\geq0}\mathrm{rank}_{O_K}O_K\langle\widehat{H}^0(\overline{\mathcal{L}}\otimes\wp^i)\rangle\cdot\log N_\wp+O(h^0(\mathcal{L}_\mathbb{Q})\log h^0(\mathcal{L}_\mathbb{Q})). \tag{2}$$

Compare (1) and (2). We have

$$\#v(\overline{\mathcal{L}})\cdot\log N_\wp=\widehat{h}^0(\overline{\mathcal{L}})+O(h^0(\mathcal{L}_\mathbb{Q})\log h^0(\mathcal{L}_\mathbb{Q})).$$

Replace $\overline{\mathcal{L}}$ by $m\overline{\mathcal{L}}$. We obtain Theorem 3.1.

Construction of Boucksom–Chen

In [BC], Boucksom and Chen constructed Okounkov bodies in a very general arithmetic setting. Here we briefly compare their construction with our construction in the setting of this paper.

We first recall the construction of [BC]. Let $(X,\overline{\mathcal{L}})$ be as above. Denote by $Z_1=X_K$ the generic fiber. Fix a flag

$$Z_1\supset Z_2\supset\cdots\supset Z_d$$

on the generic fiber Z_1. It gives an Okounkov body $\Delta_{Z.}(\mathcal{L}_K)\subset\mathbb{R}^{d-1}$ of the generic fiber $\mathcal{L}_K$.

Restricted to the generic fiber, we have a valuation

$$\nu_{Z.}^\circ=(\nu_2',\cdots,\nu_d'):\widehat{H}^0(X,\overline{\mathcal{L}})-\{0\}\to\mathbb{Z}^{d-1}.$$

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It gives a convex body $\Delta^{\circ}_{Z.}(\overline{\mathcal{L}})$ contained in $\Delta_{Z.}(\mathcal{L}_K)$.

More generally, for any $t \in \mathbb{R}$, denote $\overline{\mathcal{L}}(-t) = (\mathcal{L}, e^t\| \cdot \|)$. Then we have a convex body $\Delta^{\circ}_{Z.}(\overline{\mathcal{L}}(-t))$ of $\overline{\mathcal{L}}(-t)$ contained in $\Delta_{Z.}(\mathcal{L}_K)$. The sequence $\{\Delta^{\circ}_{Z.}(\overline{\mathcal{L}}(-t))\}_{t \in \mathbb{R}}$ is decreasing in t. Define the incidence function $G_{Z.} : \Delta_{Z.}(\mathcal{L}_K) \to \mathbb{R}$ by

$$G_{Z.}(x) = \sup\{t \in \mathbb{R} : x \in \Delta^{\circ}_{Z.}(\overline{\mathcal{L}}(-t))\}.$$

Its graph gives a $(d+1)$-dimensional convex body

$$\Delta_{Z.}(\overline{\mathcal{L}}) = \{(x,t) \in \Delta_{Z.}(\mathcal{L}_K) \times \mathbb{R} : 0 \le t \le G_{Z.}(x)\}.$$

This is the Okounkov body constructed in [BC]. One main result of [BC] asserts that

$$\mathrm{vol}(\Delta_{Z.}(\overline{\mathcal{L}})) = \frac{1}{d!}\mathrm{vol}(\overline{\mathcal{L}}).$$

Go back to our construction. Recall that we have a flag

$$X \supset Y_1 \supset \cdots \supset Y_d$$

on X. On the special fiber $Y_1 = X_{\mathbb{F}_{\wp}}$, we have a flag

$$Y_1 \supset Y_2 \supset \cdots \supset Y_d.$$

It gives a valuation of the line bundle $\mathcal{L}_{\mathbb{F}_{\wp}}$ on Y_1. Thus we get an Okounkov body $\Delta_{Y.}(\mathcal{L}_{\mathbb{F}_{\wp}})$ in $\mathbb{R}^{d-1}$.

Restricted to the special fiber, we have a valuation

$$\nu^{\circ}_{Y.} = (\nu_2, \cdots, \nu_d) : \widehat{H}^0(X, \overline{\mathcal{L}}) - \{0\} \to \mathbb{Z}^{d-1}.$$

It gives a convex body $\Delta^{\circ}_{Y.}(\overline{\mathcal{L}})$ contained in $\Delta_{Y.}(\mathcal{L}_{\mathbb{F}_{\wp}})$.

The hermitian line bundle $\overline{\mathcal{L}} \otimes \wp^t$ gives a convex body $\Delta^{\circ}_{Y.}(\overline{\mathcal{L}} \otimes \wp^t)$ in $\mathbb{R}^{d-1}$ for any $t \in \mathbb{Z}$. By passing to tensor powers, extend the definition of $\Delta^{\circ}_{Y.}(\overline{\mathcal{L}} \otimes \wp^t)$ to any $t \in \mathbb{Q}$. The sequence $\{\Delta^{\circ}_{Y.}(\overline{\mathcal{L}} \otimes \wp^t)\}_{t \in \mathbb{Q}}$ is decreasing in t. Define the incidence function $G_{Y.} : \Delta_{Y.}(\mathcal{L}_{\mathbb{F}_{\wp}}) \to \mathbb{R}$ by

$$G_{Y.}(x) = \sup\{t \in \mathbb{Q} : x \in \Delta^{\circ}_{Y.}(\overline{\mathcal{L}} \otimes \wp^t)\}.$$

Its graph gives a $(d+1)$-dimensional convex body

$$\Delta_{Y.}(\overline{\mathcal{L}}) = \{(x,t) \in \Delta_{Y.}(\mathcal{L}_{\mathbb{F}_{\wp}}) \times \mathbb{R} : 0 \le t \le G_{Y.}(x)\}.$$

It recovers our previous construction.

Hence, we see that the similarity of these two constructions. The construction of [BC] is archimedean in nature, while the construction of [Yu] is non-archimedean in nature.

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